SIGNATURE OF UNITS AND PSEUDO-UNITS

by

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Abstract. — Given a number field K with r_1 real embeddings $\sigma_1, \dots, \sigma_{r_1}$, the signature of a nonzero element x is the r_1–tuple of the signs of the $\sigma_i(x)$, represented as elements in the finite field $\mathbb{F}_2$ with two elements. We show that the pseudo-units of K take at least 2^{d_2} different signatures, with $\frac{r_1}{2} \leq d_2$. We exhibit some explicit examples to show that this lower bound does not hold for the group of units of K. We also give some numerical evidence for conjectures about the optimality of this lower bound.

***Résumé* (Signature des unités et des pseudo–unités).** — Dans un corps de nombres K ayant r_1 plongements réels $\sigma_1, \dots, \sigma_{r_1}$, la signature d'un élément non nul x est le r_1–uplet formé des signes des $\sigma_i(x)$, représentés comme éléments du corps fini $\mathbb{F}_2$ à deux éléments. Nous montrons que les pseudo–unités de K prennent au moins 2^{d_2} signatures différentes, avec $\frac{r_1}{2} \leq d_2$. Nous montrons à l'aide de quelques exemples que cette minoration n'est pas vraie si l'on considère les unités de K. À partir d'exemples numériques explicites, nous énonçons deux conjectures relatives à l'optimalité de cette minoration.

Introduction

Let $K/\mathbb{Q}$ be a number field of degree n with r_1 real embeddings. The signature of an element $x \in K^\times$ is the tuple of the signs of its r_1 real embeddings, seen as elements of the finite field $\mathbb{F}_2$: this gives a well defined group homomorphism $\mathrm{sgn} : K^\times \to (\mathbb{F}_2)^{r_1}$. Of course, the square of an element of $K^\times$ is totally positive, and we can define sgn on the quotient $K^\times/(K^\times)^2$, making sgn a linear map of vector spaces over $\mathbb{F}_2$. Several questions arise concerning sgn when restricted to a subgroup G of $K^\times$ or of $K^\times/(K^\times)^2$, among which two are classical:

Question 1. — *Is a totally positive element of G always a square? If not, what can be said about the group that measures the default?*

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Question 2. — *How many different signatures can take elements of G?*

Question 1 is the question of the injectivity of sgn. Since $K^\times/(K^\times)^2$ is an infinite dimensional $\mathbb{F}_2$–vector space, sgn is very far from being injective. Injectivity can only hold if we restrict to finite subgroups G of dimension over $\mathbb{F}_2$ at most r_1. An interesting case is the group of units (modulo squares of units), which has dimension exactly r_1 over $\mathbb{F}_2$ when the field K is totally real. The first nontrivial example is when K is a real quadratic number field: in this case, the injectivity is equivalent to the surjectivity, and is equivalent to the existence of a unit with norm -1. In [**6**], Garbanati proves that this equivalence still holds in the more general case of a real Galois extension of $\mathbb{Q}$ of degree a power of 2.

Question 2 is the question of the dimension over $\mathbb{F}_2$ of the image of sgn, and possibly of its surjectivity. The well-known Approximation Theorem asserts that sgn is surjective for $G = K^\times$. Weber [**11**] proves that if G is the group generated by the units in the real subfield of the cyclotomic field $\mathbb{Q}(\zeta_{2^k})$, then sgn is surjective, and Garbanati [**5**] considers the subgroup of circular units in real abelian extensions of $\mathbb{Q}$. In [**3**], Berger considers some totally real number fields and proves that sgn is surjective when restricted to their group of units (resp S-units for some explicit sets S). In all these situations, the dimension of $\operatorname{sgn} G$ is as large as possible, namely $\dim(\operatorname{sgn} G) = r_1$. However, as we will see in section 3, surjectivity is far from being the only possibility.

In the present paper, we address the question 2 for a general number field, and for two different groups G. The first is $G = G_1$, the group generated by the units of K; the second is $G = G_2$ the group generated by the pseudo-units. A unit is an element $x \in K^\times$ that has valuation 0 at every prime; a pseudo-unit is an element that has even valuation at every prime. The group of pseudo-units is useful to compute explicitly unramified quadratic extensions of K, via Kummer Theory (see for example [**8**]). Our main result is that the group G_2 can not have too few different signatures. More precisely:

Theorem 1. — *If G_2 is the group of pseudo-units of a number field K with signature (r_1, r_2), then the integer $\dim(\operatorname{sgn} G_2)$ can only take its values in the interval*

$$\frac{r_1}{2} \leq \dim(\operatorname{sgn} G_2) \leq r_1.$$

As noticed in [**9**], the group of all possible signatures of pseudo-units is linked to (the elements of order 2 in) the ordinary and the narrow class group of K. These groups will play a role in the proof of Theorem 1 given in section 2. Because of this link, it would not be surprising that the groups $\operatorname{sgn} G_1$ and $\operatorname{sgn} G_2$ share more properties with class groups. For example, in the same spirit as the heuristics of Cohen and Lenstra [**4**] for the class groups of number fields, is there a natural density of number fields of given signature (r_1, r_2) and groups $\operatorname{sgn} G_1$ or $\operatorname{sgn} G_2$? What is the average size of these groups ? An answer is given for the cubic case in [**2**].

Without going further in this direction, we still make the conjecture that our interval is optimal in the following sense:

***Conjecture 2* (weak form).** — *Consider a degree $n \geq 2$ and a signature (r_1, r_2) with $r_1 + 2r_2 = n$. For each integer d_2 such that $\frac{r_1}{2} \leq d_2 \leq r_1$, there exists at least one number field K of degree n and signature (r_1, r_2) such that $\dim(\operatorname{sgn} G_2) = d_2$.*

***Conjecture 3* (strong form).** — *Consider a degree $n \geq 2$ and a signature (r_1, r_2) with $r_1 + 2r_2 = n$ and $1 \leq r_1$. For each integer d_2 and each integer d_1 satisfying the inequalities*

$$\begin{array}{lll} \frac{r_1}{2} & \leq d_2 & \leq r_1 \\ 1 & \leq d_1 & \leq d_2, \end{array}$$

there exists at least one number field K of degree n and signature (r_1, r_2) such that $\dim(\operatorname{sgn} G_1) = d_1$ and $\dim(\operatorname{sgn} G_2) = d_2$.

As suggested by the anonymous referee, the questions of this paper naturally extend to the group G of S-units for a finite set S of places of K, including the places at infinity. In that case, the map sgn should probably be extended to $\prod_{\mathfrak{p} \in S} K_{\mathfrak{p}}^{\times}/(K_{\mathfrak{p}}^{\times})^2$. We leave this for future work.

This paper is organized as follows. Section 1 contains the notation and the definitions that are necessary to precisely state the theorem and the conjectures. It also contains some immediate results. The proof of Theorem 1 is given in section 2, and the computations that support the conjectures are described in section 3.

1. Preliminaries

Let $K/\mathbb{Q}$ be a number field of degree n with r_1 real embeddings $\sigma_1, \ldots, \sigma_{r_1}$, and r_2 pairs of conjugate nonreal embeddings, such that $r_1 + 2r_2 = n$. We denote by $\mathbb{F}_2$ the finite field with 2 elements and by $s : \mathbb{R}^{\times} \to \mathbb{F}_2$ the sign homomorphism where $s(x) = 0$ if $x > 0$ and $s(x) = 1$ if $x < 0$.

Composing s with $(\sigma_1, \ldots, \sigma_{r_1})$ gives the signature map $\operatorname{sgn} : K^{\times} \to (\mathbb{F}_2)^{r_1}$, which is a group homomorphism. It clearly has $(K^{\times})^2$ in its kernel, hence we also have a sgn homomorphism defined over $K^{\times}/(K^{\times})^2$. All the groups involved here have exponent 2, hence are vector spaces over $\mathbb{F}_2$. In particular, we can use the notion of dimension over $\mathbb{F}_2$, just denoted by dim.

We denote by $\mathcal{I}(K)$ the group of fractional ideals of K and by i the natural map from $K^{\times}$ to $\mathcal{I}(K)$. The kernel of i is the group $U(K)$ of units of K, and its cokernel is the class group $Cl(K)$. This is summarized by the exact sequence

$$1 \to U(K) \to K^{\times} \xrightarrow{i} \mathcal{I}(K) \to Cl(K) \to 1.$$

We define the group G_1 as the quotient of $U(K)$ by its squares:

$$G_1 \stackrel{\text{def}}{=} U(K)/U(K)^2.$$

By Dirichlet's Unit Theorem, $U(K)$ is the product of a cylic group of even order and a free group of rank $r_1 + r_2 - 1$, hence G_1 is a vector space over $\mathbb{F}_2$ of dimension $r_1 + r_2$. It is useful to see it as the subgroup of $K^{\times}/(K^{\times})^2$ generated by the units.

We also define the group G_2 as the subgroup of $K^\times/(K^\times)^2$ generated by the elements that have even valuations at all primes (pseudo-units). It can therefore be defined as

$$G_2 \overset{\text{def}}{=} \ker\left(i : K^\times/(K^\times)^2 \to \mathcal{I}(K)/\mathcal{I}(K)^2\right).$$

Since units have zero (hence even) valuation at every prime, G_1 is a subgroup of G_2. For an abelian group C, we write $C[2]$ for the subgroup of elements of order 2. The link between G_1 and G_2 is described by an exact sequence, the proof of which is left to the reader:

Proposition 4. — *The sequence*

$$1 \to G_1 \to G_2 \overset{j}{\to} Cl(K)[2] \to 1$$

is exact, where $j(x)$ is defined by the class of the ideal I such that $i(x) = I^2$.

From the previous remarks, and the fact that $-1 \in G_1$, we immediately deduce the following result.

Proposition 5. —
If $r_1 = 0$, then $\dim(\operatorname{sgn} G_1) = \dim(\operatorname{sgn} G_2) = 0$.
If $r_1 \geq 1$, then

$$1 \leq \dim(\operatorname{sgn} G_1) \leq \dim(\operatorname{sgn} G_2) \leq r_1.$$

In particular, if $r_1 = 1$, then $\dim(\operatorname{sgn} G_1) = \dim(\operatorname{sgn} G_2) = 1$.

We still need to introduce several classical groups. Let $K^+ \overset{\text{def}}{=} \ker\left(\operatorname{sgn} : K^\times \to (\mathbb{F}_2)^{r_1}\right)$ be the subgroup of totally positive elements of $K^\times$. Because sgn is surjective on $K^\times$, we have $\dim(K^\times/K^+) = r_1$. The kernel of i restricted to K^+ is the group $U(K)^+$ of totally positive units of K, and its cokernel is the narrow class group $Cl(K)^+$. This is summarized by the exact sequence

$$1 \to U(K)^+ \to K^+ \overset{i}{\to} \mathcal{I}(K) \to Cl(K)^+ \to 1.$$

2. Proof of the main result

Proposition 6. — *We have*

$$\dim(\operatorname{sgn} G_2) = r_1 - \left(\dim(Cl(K)^+[2]) - \dim(Cl(K)[2])\right).$$

Proof. — By Proposition 5, the statement is true if $r_1 = 0$, that is if the field is totally complex. We will therefore assume that $r_1 > 0$. This implies in particular that the only roots of unity in K are ± 1. We need to define two more useful groups:

$$G_+ \overset{\text{def}}{=} \ker\left(i : K^+/(K^\times)^2 \to \mathcal{I}(K)/\mathcal{I}(K)^2\right)$$
$$G'_+ \overset{\text{def}}{=} \ker\left(i : K^+/(K^+)^2 \to \mathcal{I}(K)/\mathcal{I}(K)^2\right)$$

We have $\ker\left(\operatorname{sgn} : G_2 \to (\mathbb{F}_2)^{r_1}\right) = G_+$, hence

$$\dim(\operatorname{sgn} G_2) = \dim G_2 - \dim G_+. \tag{1}$$

The groups G_+ and G'_+ are related by the exact sequence

$$1 \to \{\pm 1\} \to K^\times/K^+ \to G'_+ \to G_+ \to 1$$

where the middle arrow maps an element x to x^2. By the Approximation Theorem, we know that $\dim(K^\times/K^+) = r_1$, which gives the relation

$$\dim G_+ = \dim G'_+ - r_1 + 1. \tag{2}$$

We have already seen that $\dim G_1 = r_1 + r_2$, and by the exact sequence of Proposition 4 we obtain

$$\dim G_2 = r_1 + r_2 + \dim(Cl(K)[2]). \tag{3}$$

The corresponding exact sequence concerning the totally positive elements is

$$1 \to U(K)^+/(U(K)^+)^2 \to G'_+ \to Cl(K)^+[2] \to 1.$$

This gives $\dim G'_+ = \dim U(K)^+/(U(K)^+)^2 + \dim(Cl(K)^+[2])$. Since $-1 \notin U(K)^+$ and $U(K)^2 \subset U(K)^+$, Dirichlet's Unit Theorem asserts that $U(K)^+$ is a free submodule of $U(K)$ of rank $r_1 + r_2 - 1$, hence $\dim U(K)^+/(U(K)^+)^2 = r_1 + r_2 - 1$, and

$$\dim G'_+ = r_1 + r_2 - 1 + \dim(Cl(K)^+[2]). \tag{4}$$

Combining the relations (1) to (4), we obtain the conclusion. □

Proof of Theorem 1. — As a consequence of the 'Spiegelungsatz' of Leopoldt, it is proved in [**7**, Th. 8.12] (see also [**1**]) that the difference $\dim(Cl(K)^+[2]) - \dim(Cl(K)[2])$ is bounded by $r_1/2$. Theorem 1 then follows from Proposition 6. □

3. Numerical evidence for the conjectures

I made some experiments on the optimality of Theorem 1 that led me to the conjectures 2 and 3. All these computations were made using `pari/gp` ([**10**]).

For a finite abelian group C, with elementary divisors $[c_1, \ldots, c_k]$, the dimension over $\mathbb{F}_2$ of $C[2]$ is just the number of even c_i. In `pari/gp`, the corresponding function is

```
dim2(C) = sum(i=1,length(C),C[i]%2==0)
```

Given an irreducible polynomial P defining a number field K, we use Proposition 6 to compute the value $d_2 = \dim(\operatorname{sgn} G_2)$. In `pari/gp`, the corresponding function is

```
{d2(P) =
  bnf = bnfinit(P);
  bnr = bnrinit(bnf,[1,vector(bnf.r1,i,1)]);
  bnf.r1 - ( dim2(bnr.clgp.cyc) - dim2(bnf.clgp.cyc) )
}
```

An optimality result for Theorem 1 is formulated in Conjecture 2. As proved by the explicit examples given below, this conjecture is true for small degrees.

Proposition 7. — *Conjecture 2 is true for $n \leq 10$.*

Proof. —

n	r_1	d_2	Disc	field
2	2	1	12	$x^2 - 3$
		2	5	$x^2 + x - 1$
3	3	2	229	$x^3 + 3x^2 - x - 2$
		3	49	$x^3 + x^2 - 2x - 1$
4	2	1	-1323	$x^4 + x^3 - 3x^2 + x + 1$
		2	-275	$x^4 + x^3 - 2x - 1$
4	4	2	10512	$x^4 + 4x^3 - x^2 - 4x + 1$
		3	1125	$x^4 - 3x^3 - x^2 + 3x + 1$
		4	725	$x^4 + x^3 - 3x^2 - x + 1$

n	r_1	d_2	Disc	field
5	3	2	−32411	$x^5 + x^4 + 2x^3 - 2x^2 - 2x + 1$
		3	−4511	$x^5 - x^3 - 2x^2 + 1$
5	5	3	638597	$x^5 + 7x^4 - 2x^3 - 8x^2 + x + 2$
		4	36497	$x^5 - 2x^4 - 3x^3 + 5x^2 + x - 1$
		5	14641	$x^5 + x^4 - 4x^3 - 3x^2 + 3x + 1$
6	2	1	231173	$x^6 + x^5 - 5x^4 - 2x^3 + 9x^2 - 2x - 7$
		2	28037	$x^6 + x^5 + 2x^3 - x^2 - 3x - 1$
6	4	2	−2840383	$x^6 - 2x^5 - 3x^4 - x^3 + 2x^2 + 24x + 8$
		3	−215811	$x^6 + x^5 - 2x^4 + 2x^3 + x^2 - 3x + 1$
		4	−92779	$x^6 + x^5 - 2x^4 - 3x^3 - x^2 + 2x + 1$
6	6	3	66547629	$x^6 - 2x^5 - 10x^4 + 18x^3 + 18x^2 - 22x + 1$
		4	2415125	$x^6 + 6x^5 + 3x^4 - 12x^3 - 3x^2 + 7x - 1$
		5	453789	$x^6 + 5x^5 + 4x^4 - 8x^3 - 5x^2 + 3x + 1$
		6	300125	$x^6 + x^5 - 7x^4 - 2x^3 + 7x^2 + 2x - 1$
7	3	2	7280161	$x^7 - 5x^5 - 2x^4 + 6x^3 + 3x^2 - x - 1$
		3	612233	$x^7 - x^6 + x^5 - x^3 + x^2 - x - 1$
7	5	3	−189212591	$x^7 + 6x^6 + 5x^5 - 10x^4 - 8x^3 + 5x^2 + 3x - 1$
		4	−7211207	$x^7 - x^6 - 6x^5 + 2x^4 + 9x^3 - 3x - 1$
		5	−2306599	$x^7 - 3x^5 - x^4 + x^3 + 3x^2 + x - 1$
7	7	4	14698041649	$x^7 + 8x^6 + 8x^5 - 14x^4 - 11x^3 + 7x^2 + 3x - 1$
		5	174368473	$x^7 - 6x^6 + 7x^5 + 9x^4 - 11x^3 - 5x^2 + 3x + 1$
		6	34554953	$x^7 - 2x^6 - 5x^5 + 9x^4 + 5x^3 - 8x^2 + 1$
		7	20134393	$x^7 - x^6 - 6x^5 + 4x^4 + 10x^3 - 4x^2 - 4x + 1$
8	2	1	−65641219	$x^8 - x^7 - 5x^6 + 5x^5 + 7x^4 - 6x^3 - 3x^2 + 2x + 1$
		2	−4286875	$x^8 - 8x^7 + 30x^6 - 63x^5 + 78x^4 - 53x^3 + 13x^2 + 4x - 1$
8	4	2	1204367616	$x^8 - 6x^7 + 11x^6 + 2x^5 - 30x^4 + 32x^3 - 6x^2 - 4x + 1$
		3	50375981	$x^8 - 6x^6 - x^5 + 11x^4 + x^3 - 6x^2 + 1$
		4	15243125	$x^8 - 5x^7 + 6x^6 + 3x^5 - 15x^4 + 19x^3 - 11x^2 + 4x - 1$
8	6	3	−139132345032	$x^8 - 4x^7 + 15x^5 - 21x^4 + 5x^3 + 17x^2 - 16x + 4$
		4	−1216830647	$x^8 + 6x^7 + 7x^6 - 8x^5 - 13x^4 + 2x^3 + 6x^2 + x - 1$
		5	−74671875	$x^8 - 4x^7 + 9x^5 - 6x^4 - x^3 + 5x^2 - 4x + 1$
		6	−65106259	$x^8 - 4x^7 + x^6 + 9x^5 - x^4 - 10x^3 - 2x^2 + 4x + 1$
8	8	4	20462483441920	$x^8 + 10x^7 + 15x^6 - 38x^5 - 10x^4 + 32x^3 - 6x^2 - 4x + 1$
		5	372030717952	$x^8 + 2x^7 - 8x^6 - 10x^5 + 16x^4 + 10x^3 - 8x^2 - 2x + 1$
		6	1635340493	$x^8 + 2x^7 - 7x^6 - 9x^5 + 16x^4 + 7x^3 - 10x^2 + 1$
		7	309593125	$x^8 + 5x^7 + 2x^6 - 16x^5 - 8x^4 + 14x^3 + 5x^2 - 3x - 1$
		8	282300416	$x^8 - 2x^7 - 7x^6 + 12x^5 + 8x^4 - 14x^3 + 4x - 1$

n	r_1	d_2	Disc	field
9	3	2	-2213524327	$x^9+3x^8-2x^7-14x^6-7x^5+15x^4+15x^3-3x-1$
		3	-109880167	$x^9-2x^8+x^7+x^6-3x^5+x^4+3x^3-2x-1$
9	5	3	132798878677	$x^9-x^8-6x^7+6x^6+9x^5-12x^4-2x^3+7x^2-2x-1$
		4	2295603901	$x^9+x^8-7x^7-6x^6+16x^5+11x^4-14x^3-6x^2+4x+1$
		5	453771377	$x^9+2x^8-x^7-2x^6-x^5-5x^4+x^3+5x^2-1$
9	7	4	-76411267973087	$x^9+6x^8-8x^7-22x^6+14x^5+25x^4-6x^3-10x^2+1$
		5	-126511031459	$x^9-3x^8-4x^7+13x^6+6x^5-15x^4-5x^3+5x^2+2x-1$
		6	-2385869687	$x^9+x^8-8x^7-7x^6+21x^5+14x^4-22x^3-8x^2+8x-1$
		7	-1904081383	$x^9+x^8-6x^7-8x^6+10x^5+19x^4-2x^3-13x^2-2x+1$
9	9	5	104774245305921634708	$x^9+4x^8-11x^7-39x^6+43x^5+108x^4-63x^3-104x^2+32x+28$
		6	1370685981099672	$x^9-3x^8-7x^7+21x^6+11x^5-37x^4-x^3+16x^2-2x-1$
		7	473335756973	$x^9+2x^8-9x^7-17x^6+20x^5+30x^4-20x^3-14x^2+9x-1$
		8	16240385609	$x^9-x^8-9x^7+4x^6+26x^5-2x^4-25x^3-x^2+7x+1$
		9	9685993193	$x^9+2x^8-7x^7-14x^6+15x^5+30x^4-10x^3-19x^2+2x+1$
10	2	1	18319472593	$x^{10}+x^9+2x^8+x^7-x^6-x^5-4x^4-x^2+2x+1$
		2	799905449	$x^{10}+x^9-x^7+x^6+x^5-x^4-x^3+x-1$
10	4	2	-1325850919936	$x^{10}-3x^8-x^6+7x^4-4x^2+1$
		3	-15000291739	$x^{10}+x^9+2x^8-x^7-3x^6-5x^5-5x^4-x^3+x^2+2x+1$
		4	-3120654523	$x^{10}-2x^9+2x^8-7x^7+8x^6-3x^5+8x^4-7x^3+2x^2-2x+1$
10	6	3	70125305540625	$x^{10}-3x^9-10x^8+25x^7-46x^5+76x^4-70x^3+40x^2-15x+3$
		4	715520093041	$x^{10}-x^9-4x^8+11x^7-4x^6-12x^5+18x^4-6x^3-6x^2+5x-1$
		5	18171765053	$x^{10}+5x^9+6x^8-3x^7-3x^6+5x^5-9x^4-13x^3+5x^2+6x+1$
		6	14002335917	$x^{10}-x^9-x^8+4x^7-4x^6+8x^4-3x^3-5x^2+x+1$
10	8	4	-3407454761534464	$x^{10}+3x^8-21x^6+27x^4-10x^2+1$
		5	-18386732163379	$x^{10}-6x^9+7x^8+13x^7-35x^6+10x^5+34x^4-26x^3-x^2+5x-1$
		6	-811053191559	$x^{10}-4x^9+x^8+14x^7-21x^6+x^5+23x^4-14x^3-4x^2+5x-1$
		7	-96433540579	$x^{10}+x^9-10x^8-14x^7+15x^6+20x^5-10x^4-5x^3+3x^2-3x+1$
		8	-70952789611	$x^{10}-2x^9-6x^8+8x^7+14x^6-6x^5-16x^4-x^3+7x^2+x-1$
10	10	5	15176560115334013	$x^{10}-x^9-65x^8+65x^7+1519x^6-1519x^5-15113x^4+15113x^3$
				$+56167x^2-56167x-29369$
		6	850657970718581	$x^{10}+11x^9+35x^8+9x^7-93x^6-39x^5+93x^4+9x^3-35x^2+11x-1$
		7	18782382666752	$x^{10}+4x^9-3x^8-26x^7-9x^6+48x^5+25x^4-28x^3-10x^2+4x+1$
		8	944409985381	$x^{10}-14x^8+15x^7+31x^6-39x^5-20x^4+29x^3+x^2-6x+1$
		9	572981288913	$x^{10}-x^9-10x^8+10x^7+34x^6-34x^5-43x^4+43x^3+12x^2-12x+1$
		10	443952558373	$x^{10}+x^9-11x^8-9x^7+29x^6+16x^5-26x^4-10x^3+9x^2+2x-1$

□

In order to compute the value of $d_1 = \dim(\operatorname{sgn} G_1)$ with pari/gp, I use the following function:

```
{d1(P) =
  bnf = bnfinit(P);
  signs = apply(x->nfeltsign(bnf,x)~,concat(bnf.tu[2],bnf.fu));
  signs = matconcat(signs);
  signF2 = apply(x->(1-x)/2,signs);
  matrank(signF2*Mod(1,2))
}
```

Using this function, I could observe that it may happen that the lower bound $\frac{r_1}{2}$ is false if we just look at the signatures of the units, that is if we replace G_2 by G_1. Some examples are recorded in the next table.

n	r_1	d_1	Disc	field
3	3	1	15529	$x^3 - 19x - 21$
4	4	1	176400	$x^4 - 19x^2 + 64$
4	4	1	247104	$x^4 - 2x^3 - 24x^2 + 4x + 94$
5	3	1	-2477131	$x^5 + 5x^4 + 4x^3 - 3x^2 + 8x + 1$
5	5	1	1073025317443036	$x^5 - 6x^4 - 67x^3 + 289x^2 + 680x - 1141$
5	5	2	234799409	$x^5 + x^4 - 13x^3 - 3x^2 + 10x - 1$
6	4	1	-7110909351	$x^6 + 5x^5 - 13x^4 - 5x^3 + 55x^2 - 63x + 21$
6	6	2	566661029888	$x^6 - 20x^4 + 19x^2 - 2$
7	3	1	2132993477	$x^7 + 2x^6 - 5x^5 - 12x^4 + 2x^3 + 12x^2 + 4x - 1$
7	5	2	-26480072251759	$x^7 + 5x^6 - 26x^5 + 12x^4 + 26x^3 - 22x^2 + 4x + 1$
7	7	3	16944976450166760	$x^7 + x^6 - 25x^5 + 23x^4 + 50x^3 - 31x^2 - 20x + 5$

The same computation led me to a proof of Conjecture 3 in small degree.

Proposition 8. — *Conjecture 3 is true for $n \leq 4$.*

Proof. —

n	r_1	d_2	d_1	Disc	field
2	2	1	1	12	$x^2 - 3$
2	2	2	1	136	$x^2 - 34$
			2	5	$x^2 + x - 1$
3	3	2	1	15529	$x^3 - 19x^2 - 21$
			2	229	$x^3 + 3x^2 - x - 2$
3	3	3	1	494209	$x^3 - x^2 - 234x + 729$
			2	1957	$x^3 + 5x^2 - x - 4$
			3	49	$x^3 + x^2 - 2x - 1$
4	2	1	1	-1323	$x^4 + x^3 - 3x^2 + x + 1$
4	2	2	1	-7975	$x^4 - x^3 + 3x^2 - 12x - 16$
			2	-275	$x^4 + x^3 - 2x - 1$
4	4	2	1	176400	$x^4 - 19x^2 + 64$
			2	10512	$x^4 + 4x^3 - x^2 - 4x + 1$
4	4	3	1	6435072	$x^4 - 8x^3 - 30x^2 + 184x - 143$
			2	57600	$x^4 - 8x^2 + 1$
			3	1125	$x^4 - 3x^3 - x^2 + 3x + 1$
4	4	4	1	34830694400	$x^4 + 12x^3 - 62x^2 - 628x + 1911$
			2	6964321	$x^4 - 3x^3 - 41x^2 + 112x + 179$
			3	56025	$x^4 + 4x^3 - 3x^2 - 9x + 6$
			4	725	$x^4 + x^3 - 3x^2 - x + 1$

□

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